Computing the Hermite Form of a Matrix of Ore Polynomials

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Abstract

Let $\mathsf{F}[\partial;\sigma,\delta]$ be the ring of Ore polynomials over a field (or skew field) F, where σ is a automorphism of F and δ is a σ-derivation. Given a matrix $A \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$, we show how to compute the Hermite form H of A and a unimodular matrix U such that $UA = H$. The algorithm requires a polynomial number of operations in F in terms of n and the degrees (in ∂) of the entries in A. When $\mathsf{F} = \mathsf{k}(z)$ for some field k, it also requires time polynomial in the degree in z, and if $\mathsf{k} = \mathbb{Q}$ it requires time polynomial in the bit length of the coefficients as well. Explicit analyses are provided for the complexity, in particular for the important cases of differential and shift polynomials over $\mathbb{Q}(z)$. To accomplish our algorithm, we develop the Dieudonné determinant and quasideterminant theory for Ore polynomial rings to get explicit bounds on the degrees and sizes of entries in H and U.

1. Introduction

The Ore polynomials are a natural algebraic structure which capture difference, q-difference, differential, and other non-commutative polynomial rings. Linear algebra over these rings are important in solving the corresponding rings of differential and difference equations. The basic concepts of pseudo-linear algebra is introduced nicely in Bronstein and Petkovšek (1996); see Ore (1931) for the seminal introduction.

On the other hand, canonical forms of matrices over commutative principal ideal domains (such as $\mathbb{Z}$ or $\mathsf{F}[x]$, for a field F) have proven invaluable for both mathematical and computational purposes. One of the successes of computer algebra over the past three decades has been the development of fast algorithms for computing these canonical forms. These include triangular forms such as the Hermite form (Hermite, 1863), low

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degree forms like the Popov form (Popov, 1972), as well as the diagonal Smith form (Smith, 1861).

Canonical forms of matrices over non-commutative domains, especially rings of differential and difference operators, are also extremely useful. These have been examined at least since Dickson (1923), Wedderburn (1932), and Jacobson (1943). Recently they have found uses in control systems (Chyzak, Quadrat, and Robertz, 2005; Zerz, 2006; Halás, 2008). Computations with multidimensional linear systems over Ore algebras are nicely developed in Chyzak, Quadrat, and Robertz (2007), and a excellent implementation of many fundamental algorithms is provided in the OreModules package of Maple.

In this paper we consider canonical forms of matrices of Ore polynomials over a skew field F. Let $\sigma : \mathsf{F} \to \mathsf{F}$ be an automorphism of F and $\delta : \mathsf{F} \to \mathsf{F}$ be a σ-derivation. That is, for any $a, b \in \mathsf{F}$, $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$. We then define $\mathsf{F}[\partial; \sigma, \delta]$ as the set of usual polynomials in $\mathsf{F}[\partial]$ under the usual addition, but with multiplication defined by

$$\partial a = \sigma(a)\partial + \delta(a)$$

for any $a \in \mathsf{F}$. This is well-known to be a left (and right) principal ideal domain, with a straightforward euclidean algorithm (see Ore (1933)).

Some important cases over the field of rational functions $\mathsf{F} = \mathsf{k}(z)$ over a field k are as follows:

(1) $\sigma(z) = \mathcal{S}(z) = z + 1$ is the so-called *shift* automorphism of $\mathsf{k}(z)$, and δ identically zero on k. Then $\mathsf{k}(z)[\partial; \mathcal{S}, 0]$ is generally referred to as the ring of *shift polynomials*. With a slight abuse of notation we write $\mathsf{k}(z)[\partial; \mathcal{S}]$ for this ring.

(2) $\delta(z) = 1$ and $\sigma(z) = z$, so $\delta(h(z)) = h'(z)$ for any $h \in \mathsf{k}(z)$ with h' its usual derivative. Then $\mathsf{k}(z)[\partial; \sigma, \delta]$ is called the ring of *differential polynomials*. With a slight abuse of notation we write $\mathsf{k}(z)[\partial; \prime]$ for this ring.

A primary motivation in the definition of $\mathsf{k}(z)[\partial; \prime]$ is that there is a natural action on the space of infinitely differentiable functions in z, namely the differential polynomial

$$a_m\partial^m + a_{m-1}\partial^{m-1} + \cdots + a_1\partial + a_0 \in \mathsf{k}(z)[\partial; \prime]$$

acts as the linear differential operator

$$a_m(z)\frac{d^m f(z)}{dz^m} + a_{m-1}(z)\frac{d^{m-1} f(z)}{dz^{m-1}} + \cdots + a_1(z)\frac{df(z)}{dz} + a_0(z)f(z)$$

on a differentiable function $f(z)$. See Bronstein and Petkovšek (1996). Solving and analyzing systems of such operators involves working with matrices over $\mathsf{k}(z)[\partial; \prime]$, and invariants such as the differential analogues of the Smith, Popov and Hermite forms provide important structural information.

A matrix $H \in \mathsf{F}[\partial; \sigma, \delta]^{n \times n}$ of full left row rank is said to be in *Hermite form* if H is upper triangular, if every diagonal entry is monic, and every off-diagonal entry has degree less than the diagonal entry below it. For example, in the differential polynomials $\mathbb{Q}(z)[\partial; \prime]$ as above:

$$A = \begin{pmatrix} 1 + (z+2)\partial + \partial^2 & 2 + (2z+1)\partial & 1 + (1+z)\partial \\ (2z+z^2) + z\partial & (2+2z+2z^2) + \partial & 4z+z^2 \\ (3+z) + (3+z)\partial + \partial^2 & (8+4z) + (5+3z)\partial + \partial^2 & (7+8z) + (2+4z)\partial \end{pmatrix} \in \mathbb{Q}(z)[\partial; \prime]^{3\times 3} \quad (1.1)$$

has Hermite form

$$H = \begin{pmatrix} (2+z)+\partial & 1+2z & \frac{-2+z+2z^2}{2z} - \frac{1}{2z}\partial \\ 0 & (2+z)+\partial & 1 + \frac{7z}{2} + \frac{1}{2}\partial \\ 0 & 0 & -\frac{2}{z} + \frac{-1+2z+z^2}{z}\partial + \partial^2 \end{pmatrix} \in \mathbb{Q}(z)[\partial;\prime]^{3\times 3}.$$

Note that the Hermite form may have denominators in z. Also, while this example does not demonstrate it, it is common that the degrees in the Hermite form, in both z and ∂, are substantially larger than in the input.

For any matrix $A \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ of full left row rank, there exists a unique unimodular matrix $U \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ (i.e., a matrix whose inverse exists and is also in $\mathsf{F}[\partial;\sigma,\delta]^{n\times n}$) such that $UA = H$ is in Hermite form. This form is canonical in the sense that if two matrices $A, B \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ are such that $A = PB$ for unimodular $P \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ then the Hermite form of A equals the Hermite form of B. The existence and uniqueness of this form over $\mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ is established much as it is over $\mathbb{Z}^{n\times n}$, with the division with remainder and the euclidean algorithm replaced by right division with remainder and the right euclidean algorithm respectively. See, for example, Newman (1972), Theorem II.2.

In commutative domains such as $\mathbb{Z}$ and $\mathsf{F}[x]$ there have been enormous advances in the past two decades in computing Hermite, Smith and Popov forms. Polynomial-time algorithms for the Smith and Hermite forms over $\mathsf{F}[x]$ were developed by Kannan (1985), with important advances by Kaltofen, Krishnamoorthy, and Saunders (1987), Villard (1995), Mulders and Storjohann (2003), and many others. One of the key features of this recent work in computing canonical forms has been a careful analysis of the complexity in terms of matrix size, entry degree, and coefficient swell. Clearly identifying and analyzing the cost in terms of all these parameters has led to a dramatic drop in both theoretical and practical complexity.

Computing the classical Smith and Hermite forms of matrices over Ore domains has received less attention though canonical forms of differential polynomial matrices have applications in solving differential systems and control theory (see Halás (2008); Kotta, Leiback, and Halás (2008)). Abramov and Bronstein (2001) analyzes the number of reduction steps necessary to compute a *row-reduced form*, while Beckermann, Cheng, and Labahn (2006) analyze the complexity of row reduction in terms of matrix size, degree and the sizes of the coefficients of some shifts of the input matrix. Beckermann et al. (2006) demonstrates tight bounds on the degree and coefficient sizes of the output, which we will employ here. For the Popov form, Cheng (2003) gives an algorithm for matrices of shift polynomials. Cheng's approach involves order bases computation in order to eliminate lower order terms of Ore polynomial matrices. A main contribution of Cheng (2003) is to give an algorithm computing the left row rank and a row-reduced basis of the left nullspace of a matrix of Ore polynomials in a fraction-free way. This idea is extended in Davies, Cheng, and Labahn (2008) to compute Popov form of general Ore polynomial matrices. They reduce the problem of computing Popov form to a nullspace computation. However, though Popov form is useful for rewriting high order terms with respect to low order terms, we want a different canonical form more suited to solving system of linear diophantine equations. Since the Hermite form is upper triangular, it meets this goal nicely, not to mention the fact that it is a "classical" canonical form. An implementation of the basic (exponential-time) Hermite algorithm is provided by Culianez (2005). In Giesbrecht and Kim (2009), we present a polynomial-time algorithm for the Hermite

form over $\mathbb{Q}(t)[\partial;\prime]$. While it relies on similar techniques as this current paper, the cost of the algorithm is considerably higher, and the coefficient bounds weaker, as well as not obviously working for all Ore polynomials.

The related "two-sided" problem of computing the Jacobson (non-commutative Smith) canonical form has also been recently considered. Blinkov, Cid, Gerdt, Plesken, and Robertz (2003) implement the standard algorithm in Janet. Levandovskyy and Schindelar (2011) provide a very complete implementation, for the full Ore case over skew fields, of a Jacobson form algorithm using Gröbner bases in Singular. Middeke (2008) has recently demonstrated for the Jacobson form of a matrix of differential polynomials, which requires time polynomial in the matrix size and degree (but the coefficient size is not analyzed).

One of the primary difficulties in both developing efficient algorithms for matrices of Ore polynomials, and in their analysis, is the lack of a standard determinant, and the important bounds this provides on degrees in eliminations. In Section 2 we establish bounds on the degrees of entries in the inverse of a matrix over any non-commutative field with a reasonable degree function. We do this by introducing the *quasideterminant* of Gel'fand and Retakh (1991, 1992) and analyzing its interaction with the degree function. We also prove similar bounds on the degree of the Dieudonné determinant. In both cases, the bounds are essentially the same as for matrices over a commutative function field.

In Section 3 we consider matrices over the Ore polynomials and bound the degrees of entries in the Hermite form and corresponding unimodular transformation matrices. We also bound the degrees of the Dieudonné determinants of these matrices.

In Section 4 we give an algorithm that, given a matrix $A \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ (of full left row rank), computes H and U such that $UA = H$, which requires a polynomial number of operations in n, $\deg A$, and the size of the coefficients of the entries of A. In the common case where $\mathsf{F} = \mathbb{Q}(z)$ these algorithms require time polynomial in n, $\deg_\partial(A)$, $\deg_z(A)$, and the bit-length of the rational polynomial coefficients. Our algorithms could no doubt be generalized to the non-square and non-full-rank cases, but we do not pursue this here. We also give explicit upper bounds on the degrees and coefficient sizes of U and H.

2. Non-commutative determinants and degree bounds for linear equations

One of the main difficulties in matrix computations in skew (non-commutative) fields, and a primary difference with the commutative case, is the lack of the usual determinant. In particular, the determinant allows us to bound the degrees of solutions to systems of equations, the size of the inverse or other decomposition, not to mention the degrees at intermediate steps of computations, through Hadamard-like formulas and Cramer's rules.

The most common non-commutative determinant was defined by Dieudonné (1943), and is commonly called the *Dieudonné determinant*. It preserves some of the multiplicative properties of the usual commutative determinant, but is insufficient to establish the degree bounds we require (amongst other inadequacies). In Gel'fand and Retakh (1991, 1992) introduced *quasideterminants* and a rich associated theory as a central tool in linear algebra over non-commutative rings. Quasideterminants are more akin to the (inverse of the) entries of the classical adjoint of a matrix than a true determinant. We employ quasideterminants here to establish bounds on the degree of the entries in the inverse of a matrix, and on the Dieudonné determinant in this section, and on the Hermite form and its multiplier matrices in Section 3

We will establish bounds on degrees of quasideterminants and Dieudonné determinants for a general skew field K with a *degree function* $\deg : \mathsf{K} \to \mathbb{Z} \cup \{-\infty\}$ satisfying the following. For $a, b \in \mathsf{K}$:

(i) If $a \neq 0$ then $\deg a \in \mathbb{Z}$, and $\deg 0 = -\infty$;
(ii) $\deg(a+b) \leq \max\{\deg a, \deg b\}$;
(iii) $\deg(ab) = \deg a + \deg b$;
(iv) If $a \neq 0$ then $\deg(a^{-1}) = -\deg a$.

For example, if $\mathsf{K} = \mathsf{F}(z)$ for some commutative field F and indeterminate t, for any $a \in \mathsf{K}^*$ we can define $\deg a = \deg a_N - \deg a_D$, where $a_N, a_D \in \mathsf{F}[x]$ are such that $a = a_N/a_D$.

2.1. Quasideterminants and degree bounds

Following Gel'fand and Retakh (1991, 1992), we define the *quasideterminant* as a collection of n^2 functions from $\mathsf{K}^{n\times n} \to \mathsf{K} \cup \{\perp\}$, where $\perp$ represents the function being *undefined*. Let $A \in \mathsf{K}^{n\times n}$ and $p, q \in \{1, \ldots, n\}$. Assume $A_{pq} \in \mathsf{K}$ is the (p,q) entry of K, and let $A^{(pq)} \in \mathsf{K}^{(n-1)\times(n-1)}$ be the matrix A with the pth row and qth column removed. Define the (p,q)-quasideterminant of A as

$$|A|_{pq} = A_{pq} - \sum_{i\neq p, j\neq q} A_{pi}(|A^{(pq)}|_{ji})^{-1} A_{jq},$$

where the sum is taken over all summands where $|A^{(pq)}|_{ji}$ is defined. If all summands have $|A^{(pq)}|_{ji}$ undefined then $|A|_{pq}$ is undefined (and has value $\perp$). See Gel'fand and Retakh (1992).

Fact 2.1 (Gel'fand and Retakh (1991), Theorem 1.6)**.** *Let $A \in \mathsf{K}^{n\times n}$ over a (possibly skew) field K.*

(1) The inverse matrix $B = A^{-1} \in \mathsf{K}^{n\times n}$ exists if and only if the following are true:
  (a) If the quasideterminant $|A|_{ij}$ is defined then $|A|_{ij} \neq 0$, for all $i, j \in \{1, \ldots, n\}$;
  (b) For all $p \in \{1, \ldots, n\}$ there exists a $q \in \{1, \ldots, n\}$, such that the quasideterminant $|A|_{pq}$ is defined;
  (c) For all $q \in \{1, \ldots, n\}$ there exists a $p \in \{1, \ldots, n\}$ such that the quasideterminant $|A|_{pq}$ is defined;

(2) If the inverse B exists, then for $i, j \in \{1, \ldots, n\}$ we have

$$B_{ji} = \begin{cases} (|A|_{ij})^{-1} & \text{if } |A|_{ij} \text{ is defined,} \\ 0 & \text{if } |A|_{ij} \text{ is not defined.} \end{cases}$$

We now bound the size of the quasideterminants in terms of the size of the entries of A. Assume that K has a degree function as above.

Theorem 2.2. *Let $A \in \mathsf{K}^{n\times n}$, such that either $A_{ij} = 0$ or $0 \leq \deg A_{ij} \leq d$ for all $i, j \in \{1, \ldots, n\}$. For all $p, q \in \{1, \ldots, n\}$ such that $|A|_{pq}$ is defined we have $-(n-1)d \leq |A|_{pq} \leq nd$.*

Proof. We proceed by induction on n.

For $n = 1$, $p = q = 1$ and $|A|_{11} = A_{11}$, so clearly the property holds. Assume the statement is true for dimension $n-1$. Then

$$\deg |A|_{pq} = \deg \left(A_{pq} - \sum_{i \neq p, j \neq q} A_{pi} (|A^{(pq)}|_{ji})^{-1} A_{jq} \right),$$

where the sum is over all defined summands. Then using the inductive hypothesis we have

$$\begin{aligned} \deg |A|_{pq} &\leq \max \left\{ \deg A, \max_{i \neq p, j \neq q} \left\{ \deg A_{pi} - \deg |A^{(pq)}|_{ji} + \deg A_{jq} \right\} \right\} \\ &\leq 2d + (n-2)d \leq nd, \end{aligned}$$

and

$$\deg |A|_{pq} \geq -\deg |A^{(pq)}|_{ji} \geq -(n-1)d. \qquad \square$$

Corollary 2.3. *Let $A \in \mathsf{K}^{n \times n}$ be unimodular, and $B \in \mathsf{K}^{n \times n}$ such that $AB = I$. Assume $A_{ij} = 0$ or $0 \leq \deg A_{ij} \leq d$ for all $i, j \in \{1, \ldots, n\}$. Then $\deg B \leq (n-1)d$.*

Proof. From Fact 2.1 we know that $B_{ji} = (|A|_{ij})^{-1}$ when $|A|_{ij}$ is defined (and $B_{ji} = 0$ otherwise). Thus $\deg B_{ji} = -\deg |A|_{ij} \leq (n-1)d$, and $B_{ij} = 0$ or $\deg B_{ij} \geq 0$ since A is unimodular. $\square$

2.2. Dieudonné Determinants

Let $[\mathsf{K}^*, \mathsf{K}^*]$ be the *commutator subgroup* of the multiplicative group K^* of K, the (normal) subgroup of K^* generated by all pairs of elements of the form $a^{-1}b^{-1}ab$ for $a, b \in \mathsf{K}^*$. Thus $\mathsf{K}^*/[\mathsf{K}^*, \mathsf{K}^*]$ is a commutative group.

Let $A \in \mathsf{K}^{n \times n}$ be a matrix with a right inverse. The *Bruhat Normal Form* of A is a decomposition $A = TDPV$, where $P \in \mathsf{K}^{n \times n}$ is a permutation matrix inducing the permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$, and $T, D, V \in \mathsf{K}^{n \times n}$ are

$$T = \begin{pmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \qquad D = \operatorname{diag}(u_1, \ldots, u_n), \qquad V = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ * & \cdots & * & 1 \end{pmatrix}.$$

The Bruhat decomposition arises from gaussian elimination, much as the LUP decomposition does in the commutative case. We then define $\delta\varepsilon\tau(A) = \operatorname{sign}(\sigma) \cdot u_1 \cdots u_n \in \mathsf{K}$ (sometimes called the *pre-determinant* of A). Let π be the canonical projection from $\mathsf{K}^* \to \mathsf{K}/[\mathsf{K}^*, \mathsf{K}^*]$. Then the Dieudonné determinant is defined as $\mathcal{D}\mathrm{et}(A) = \pi(\delta\varepsilon\tau(A)) \in \mathsf{K}/[\mathsf{K}^*, \mathsf{K}^*]$, or $\mathcal{D}\mathrm{et}(A) = 0$ if A is not invertible.

The Dieudonné determinant has a number of the desireable properties of the usual determinant, as proven in Dieudonné (1943):

(1) $\mathcal{D}\mathrm{et}(AB) = \mathcal{D}\mathrm{et}(A)\,\mathcal{D}\mathrm{et}(B)$ for any $A, B \in \mathsf{K}^{n \times n}$;

(2) $\mathcal{D}\mathrm{et}(P) = 1$ for any permutation matrix;

(3) $\mathcal{D}\mathrm{et}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \mathcal{D}\mathrm{et}(A)\,\mathcal{D}\mathrm{et}(B)$.

Also note that if K has a degree function as above, then $\deg(\mathcal{D}\mathrm{et}(A))$ is well defined, since all elements of the equivalence class of $\pi(\mathcal{D}\mathrm{et}(A))$ have the same degree (since the degree of all members of the commutator is zero. Gel'fand and Retakh (1991) show that

$$\begin{aligned}\delta\varepsilon\tau(A) &= |A|_{11}|A^{(11)}|_{22}|A^{(12,12)}|_{33}|A^{(123,123)}|_{44}\cdots|A^{(1\ldots n-1,1\ldots,n-1)}|_{nn}\\ &= |A|_{11}\cdot\delta\varepsilon\tau(A^{(11)}),\end{aligned}$$

when all these quasideterminants are defined (or equivalently P is the identity in the Bruhat decomposition above), where $A^{(1\ldots k,1\ldots k)}$ is the matrix A with rows $1\ldots k$ and columns $1\ldots k$ removed (keeping the original labelings of the remaining rows and columns).

More generally, let $R=(r_1,\ldots,r_n)$, $C=(c_1,\ldots,c_n)$ be permutations of $\{1,\ldots,n\}$, let $R_k=(r_1,\ldots,r_k)$, $C_k=(c_1,\ldots,c_k)$, and define $A^{(R_k,C_k)}$ as the matrix A with rows $r_1,\ldots,r_k$ and columns $c_1,\ldots,c_k$ removed (where $A^{(R_0,C_0)}=A$). Define

$$\begin{aligned}\delta\varepsilon\tau_{R,C}(A) &= |A|_{r_1,c_1}|A^{(R_1,C_1)}|_{r_2,c_2}|A^{(R_2,C_2)}|_{r_3,c_3}\cdot|A^{(R_{n-1},C_{n-1})}|_{r_n,c_n}\\ &= |A|_{r_1,c_1}\cdot\delta\varepsilon\tau_{R,C}(A^{(r_1,c_1)})\\ &= |A|_{r_1,c_1}\cdot|A^{(R_1,C_1)}|_{r_2,c_2}\cdot\delta\varepsilon\tau(A^{(R_2,C_2)}).\end{aligned}\tag{2.1}$$

Fact 2.4 (Gelfand, Gelfand, Retakh, and Wilson (2005), Section 3.1)**.** *Let R,C be permutations of $\{1,\ldots,n\}$ and R_k,C_k defined as above. If $|A^{(R_k,C_k)}|_{r_{k+1},c_{k+1}}$ defined for $k=0\ldots n-1$, then*

$$\mathcal{D}\mathrm{et}(A)=\mathrm{sign}(R)\cdot\mathrm{sign}(C)\cdot\pi(\delta\varepsilon\tau_{R,C}(A)).$$

In other words, the Dieudonné determinant is essentially invariant of the order of the sequence of submatrices specified in (2.1).

Theorem 2.5. *Let $A\in\mathsf{K}^{n\times n}$ be invertible, with $\deg A_{ij}\le d$. Then $\deg\mathcal{D}\mathrm{et}(A)\le nd$.*

Proof. We proceed by induction on n. For $n=1$ this is clear. For $n=2$, the possible predeterminants are

$$\begin{aligned}\delta\varepsilon\tau_{12,12}(A) &= |A|_{11}A_{22} = (A_{11}-A_{12}A_{22}^{-1}A_{21})A_{22},\\ \delta\varepsilon\tau_{12,21}(A) &= |A|_{12}A_{21} = (A_{12}-A_{11}A_{21}^{-1}A_{22})A_{21},\\ \delta\varepsilon\tau_{21,12}(A) &= |A|_{21}A_{22} = (A_{21}-A_{22}A_{12}^{-1}A_{11})A_{12},\\ \delta\varepsilon\tau_{21,21}(A) &= |A|_{22}A_{11} = (A_{22}-A_{21}A_{11}^{-1}A_{12})A_{11},\end{aligned}$$

at least one of which must be defined and non-zero, and all of which clearly have degree at most $2d$.

Now assume the theorem is true for matrices of dimension less than n. Choose $r_1,c_1\in\{1,\ldots,n\}$ such that $|A|_{r_1,c_1}$ is non-zero and of minimal degree; that is $\deg|A|_{r_1,c_1}\le\deg|A|_{k,\ell}$ for all k,ℓ such that $|A|_{k,\ell}$ defined and non-zero. The fact that $|A|_{r_1,c_1}\neq 0$ implies that $A^{(r_1,c_1)}$ is invertible, and we can continue this process recursively. Thus, let $R=(r_1,\ldots,r_n)$ and $C=(c_1,\ldots,c_n)$ be permutations of $\{1,\ldots,n\}$ such that

$|A^{(R_i,C_i)}|_{r_{i+1},c_{i+1}} \neq 0$ and $\deg |A^{(R_i,C_i)}|_{r_{i+1},c_{i+1}}$ is minimal over the degrees of non-zero, defined quasideterminants $|A^{(R_i,C_i)}|_{k,\ell}$, for $0 \leq i < n$. Now

$$\begin{aligned}
\delta\varepsilon\tau_{R,C}(A) &= \left(|A|_{r_1,c_1} \cdot |A^{(r_1,c_1)}|_{r_2,c_2} \cdot \delta\varepsilon\tau(A^{(R_2,C_2)})\right) \\
&= \left(A_{r_1,c_1} - \sum_{k,\ell} A_{r_1 k} |A^{(r_1,c_1)}|^{-1}_{\ell k} A_{\ell c_1}\right) \cdot |A^{(r_1,c_1)}|_{r_2,c_2} \cdot \delta\varepsilon\tau(A^{(R_2,C_2)}) \\
&= A_{r_1,c_1} \cdot |A^{(r_1,c_1)}|_{r_2,c_2} \cdot \delta\varepsilon\tau(A^{(R_2,C_2)}) \\
&\quad - \sum_{k,\ell} A_{r_1 k} |A^{(r_1,c_1)}|^{-1}_{\ell k} A_{\ell c_1} \cdot |A^{(r_1,c_1)}|_{r_2,c_2} \cdot \delta\varepsilon\tau(A^{(R_2,C_2)}) \\
&= A_{r_1,c_1} \cdot \delta\varepsilon\tau(A^{(R_1,C_1)}) \\
&\quad - \sum_{k,\ell} A_{r_1 k} |A^{(r_1,c_1)}|^{-1}_{\ell k} A_{\ell c_1} \cdot |A^{(r_1,c_1)}|_{r_2,c_2} \cdot \delta\varepsilon\tau(A^{(R_2,C_2)}),
\end{aligned}$$

where all sums are taken only over defined quasideterminants as above. Thus

$$\deg \mathcal{D}\mathrm{et}(A) = \deg \delta\varepsilon\tau_{R,C}(A) \leq \max\{d + (n-1)d, 2d + (n-2)d\} \leq nd,$$

using the induction hypothesis and the assumption that $\deg |A^{(r_1,c_1)}|_{r_2,c_2}$ is chosen to be minimal. □

3. Degree bounds on matrices over $\mathsf{F}[\partial; \sigma, \delta]$

Some well-known properties of $\mathsf{F}[\partial; \sigma, \delta]$ are worth recalling; see Ore (1933) for the original theory or Bronstein and Petkovšek (1994) for an algorithmic presentation. Given $f, g \in \mathsf{F}[\partial; \sigma, \delta]$, there is a degree function (in ∂) which satisfies the usual properties: $\deg(fg) = \deg f + \deg g$ and $\deg(f+g) \leq \max\{\deg f, \deg g\}$. We set $\deg 0 = -\infty$.

$\mathsf{F}[\partial; \sigma, \delta]$ is a left (and right) principal ideal ring, which implies the existence of a right (and left) division with remainder algorithm such that there exists unique $q, r \in \mathsf{F}[\partial; \sigma, \delta]$ such that $f = qg + r$ where $\deg(r) < \deg(g)$. This allows for a right (and left) euclidean-like algorithm which shows the existence of a greatest common right divisor, $h = \mathrm{gcrd}(f, g)$, a polynomial of minimal degree (in ∂) such that $f = uh$ and $g = vh$ for $u, v \in \mathsf{F}[\partial; \sigma, \delta]$. The GCRD is unique up to a left multiple in $\mathsf{F}(t)\backslash\{0\}$, and there exist co-factors $a, b \in \mathsf{F}[\partial; \sigma, \delta]$ such that $af + bg = \mathrm{gcrd}(f, g)$. There also exists a least common left multiple $\mathrm{lclm}(f, g)$. Analogously there exists a greatest common left divisor, $\mathrm{gcld}(f, g)$, and least common right multiple, $\mathrm{lcrm}(f, g)$, both of which are unique up to a right multiple in F. From Ore (1933) we also have that

$$\begin{aligned}
\deg \mathrm{lclm}(f, g) &= \deg f + \deg g - \deg \mathrm{gcrd}(f, g), \\
\deg \mathrm{lcrm}(f, g) &= \deg f + \deg g - \deg \mathrm{gcld}(f, g).
\end{aligned} \tag{3.1}$$

It will be useful to work in the quotient skew field of $\mathsf{F}(\partial; \sigma, \delta)$ of $\mathsf{F}[\partial; \sigma, \delta]$, and to extend the degree function deg appropriately. We first show that any element of $\mathsf{F}(\partial; \sigma, \delta)$ can be written as a *standard fraction* fg^{-1}, for $f, g \in \mathsf{F}[\partial; \sigma, \delta]$ (and in particular, since $\mathsf{F}[\partial; \sigma, \delta]$ is non-commutative, we insist that g^{-1} is on the right).

Fact 3.1 (see Ore (1933), Section 3)**.** *Every element of* $\mathsf{F}(\partial;\sigma,\delta)$ *can be written as a standard fraction.*

The notion of degree extends naturally to $\mathsf{F}[\partial;\sigma,\delta]$ as follows. For $f,g\in\mathsf{F}[\partial;\sigma,\delta]$, $g\neq 0$, we define $\deg(fg^{-1})=\deg f-\deg g$ for $f,g\in\mathsf{F}[\partial;\sigma,\delta]$. The proof of the next lemma is left to the reader.

Lemma 3.2. *For* $f,g,u,v\in\mathsf{F}[\partial;\sigma,\delta]$*, with* $g,v\neq 0$*, we have the following:*
(a) if $fg^{-1}=uv^{-1}$ *then* $\deg(fg^{-1})=\deg(uv^{-1})$*;*
(b) $\deg((fg^{-1})\cdot(uv^{-1}))=\deg(fg^{-1})+\deg(uv^{-1})$*;*
(c) $\deg(fg^{-1}+uv^{-1})\leq\max\{\deg(fg^{-1}),\deg(uv^{-1})\}$*;*
(d) $\deg((fg^{-1})^{-1})=-\deg(fg^{-1})$*.*

In summary, the degree function on $\mathsf{F}(\partial;\sigma,\delta)$ meets the requirement of a degree function on a skew field as in Section 2.

3.1. Determinantal degree and unimodularity

We now characterize a matrix being unimodular as those such that the degree of the Dieudonné determinant is zero.

The following Fact forms the basis for the (theoretical) reduction of a matrix to Hermite form.

Fact 3.3 (Jacobson (1943), Section 3.7)**.** *Let* $a,b\in\mathsf{F}[\partial;\sigma,\delta]$*, not both zero with* $g=\operatorname{gcrd}(a,b)$*,* $u,v\in\mathsf{F}[\partial;\sigma,\delta]$ *such that* $ua+vb=g$*, and* $s,t\in\mathsf{F}[\partial;\sigma,\delta]$ *such that* $sa=-tb=\operatorname{lclm}(a,b)$*. Then*

$$W=\begin{pmatrix}u & v\\ s & t\end{pmatrix}\in\mathsf{F}[\partial;\sigma,\delta]^{2\times 2}\quad\text{such that}\quad W\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}g\\ 0\end{pmatrix},$$

and W *us unimodular.*

Lemma 3.4. *Let* W *be as in Fact 3.3. Then* $\deg\mathcal{D}\det W=0$*.*

Proof. We may assume that $\operatorname{gcrd}(a,b)=g=1$, since the same matrix satisfies $W(ag^{-1},bg^{-1})^T=(1,0)^T$. Also assume both $ab\neq 0$ (otherwise the lemma is trivial). Then

$$\begin{pmatrix}u & v\\ s & t\end{pmatrix}\begin{pmatrix}a & 0\\ b & 1\end{pmatrix}=\begin{pmatrix}1 & v\\ 0 & t\end{pmatrix},\qquad\text{and}\quad\mathcal{D}\det(W)\cdot a\equiv t\bmod[\mathsf{F}[\partial;\sigma,\delta]^*,\mathsf{F}[\partial;\sigma,\delta]^*],$$

so $\deg\det W+\deg a=\deg t$. Since $\operatorname{gcrd}(a,b)=1$, from (3.1) we know $\deg a=\deg t$, so $\deg\det W=0$. □

Theorem 3.5. *Let* $U\in\mathsf{F}[\partial;\sigma,\delta]^{n\times n}$*. Then* U *is unimodular if and only if* $\deg\mathcal{D}\det U=0$*.*

Proof. The Hermite form can be constructed (inefficiently) by continually zeroing column entries with *elementary* unimodular matrices. Let $w=(w_1,\ldots,w_n)^T\in\mathsf{F}[\partial;\sigma,\delta]^{n\times 1}$,

and i, j distinct indices such that $w_i, w_j \neq 0$. Let W be as in Fact 3.3 with $(a, b) = (w_i, w_j)^T$, and

$$E = E(i, j; u, v, s, t) \in \mathsf{F}[\partial; \sigma, \delta]^{n \times n}$$

which is the identity matrix, except that

$$E_{ii} = u, \quad E_{ij} = v, \quad E_{ji} = s, \quad E_{jj} = t,$$

where $u, v, s, t \in \mathsf{F}[\partial; \sigma, \delta]$ are as in Fact 3.3. Then E is unimodular, and $Uw = (p_1, \ldots, p_n) \in \mathsf{F}[\partial; \sigma, \delta]^{n \times 1}$, with $p_i = \operatorname{gcrd}(w_i, w_j)$ and $p_j = 0$. By Lemma 3.4 and the properties of Dieudonné determinants, $\deg \mathcal{D}\mathrm{et}\, E = 0$. Finally, since the Hermite form of any unimodular matrix $U \in \mathsf{F}[\partial; \sigma, \delta]^{n \times n}$ is the identity, any unimodular matrix is the product of a finite number of elementary unimodular matrices, and $\deg \mathcal{D}\mathrm{et}\, U = 0$.

The converse is straightforward: if $\deg \mathcal{D}\mathrm{et}\, U = 0$, then the Hermite form is the identity matrix, and the multiplier for the Hermite gives the inverse of U. □

3.2. Degree bounds on the Hermite form

In this section we establish degree bounds on Hermite forms of matrices over $\mathsf{F}[\partial; \sigma, \delta]$ and their unimodular transformation matrices.

Theorem 3.6. *Let $A \in \mathsf{F}[\partial; \sigma, \delta]^{n \times n}$ have full left row rank and entries of degree at most d and Hermite form $H \in \mathsf{F}[\partial; \sigma, \delta]^{n \times n}$. Then*

(a) The sum of the degrees of the diagonal entries of H has degree at most nd;
(b) The sum of the degrees of the entries in any row of H has degree at most nd.

Proof. Let $V \in \mathsf{F}[\partial; \sigma, \delta]^{n \times n}$ be unimodular such that $A = VH$, whence $\mathcal{D}\mathrm{et}(A) = \mathcal{D}\mathrm{et}(V)\, \mathcal{D}\mathrm{et}(H)$. Therefore (a) follows from

$$\deg \mathcal{D}\mathrm{et}(A) = \deg \mathcal{D}\mathrm{et}(H) = \sum_{1 \leq i \leq n} \deg H_{ii} \leq nd.$$

Point (b) follows from the fact that each entry above the diagonal in the Hermite form has, by definition, degree smaller than the degree of the diagonal entry below it. □

We now show that all entries in H^{-1} have non-positive degrees.

Lemma 3.7. *Let $H \in \mathsf{F}[\partial; \sigma, \delta]^{n \times n}$ be of full left row rank and in Hermite form, and let $J = H^{-1}$. Then $\deg J_{ij} \leq 0$ for $1 \leq i, j \leq n$.*

Proof. We consider the equation $JH = I$, and note that J, like H is upper triangular. For each $r \in \{1, \ldots, n\}$ we show by induction on c (for $r \leq c \leq n$) that $\deg J_{rc} \leq 0$.

For the base case $c = r$, $J_{rr} H_{rr} = 1$, so $\deg J_{rr} = -\deg H_{rr} \leq 0$.

Assume now that $r < c$ and $\deg J_{r\ell} \leq 0$ for $r \leq \ell < c$. We need to show that $\deg J_{rc} \leq 0$. We know that

$$\sum_{1 \leq i \leq n} J_{r\ell} H_{\ell c} = \sum_{r \leq \ell \leq c} J_{r\ell} H_{\ell c} = 0.$$

Since $\deg J_{r\ell} \leq 0$ for $r \leq \ell < c$ and $\deg H_{cc} > \deg H_{\ell c}$ for $r \leq \ell < c$, it must be the case that $\deg J_{rc} \leq 0$ as well. □

Theorem 3.8. *Let $A \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ be a invertible (over $\mathsf{F}(\partial;\sigma,\delta)$), whose entries all have degree at most d. Suppose A has Hermite form $H \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$, with $UA = H$ and $UV = I$ for $U, V \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$. Then $\deg V \le d$ and $\deg U \le (n-1)\deg A$.*

Proof. Note that $V = AH^{-1}$, and by Lemma 3.7 all entries in H^{-1} have non-positive degree. Thus $\deg V \le \deg A$. By Corollary 2.3, $\deg U \le (n-1)\deg A$. □

4. Computing Hermite forms by linear systems over $\mathsf{F}[\partial;\sigma,\delta]$

In this section we present our polynomial-time algorithm to compute the Hermite form of a matrix over $\mathsf{F}[\partial;\sigma,\delta]$. This generally follows the "linear systems" approach of Kaltofen et al. (1987), and more specifically the refinements in Storjohann (1994) (for matrices over $\mathsf{k}[x]$ for a field k). We will need the tools for $\mathsf{F}[\partial;\sigma,\delta]$ we have developed in the previous sections. The method only directly constructs the matrix U such that $H = UA$. The Hermite form H can be found by performing the multiplication UA.

Assume that $A_{ij} = \sum_{0\le k\le d} A_{ijk}\partial^k$ for $A_{ijk} \in \mathsf{F}$. Let $\mathrm{row}(A,i) \in \mathsf{F}[\partial;\sigma,\delta]^{1\times n}$ be the ith row of A and define

$$\mathcal{L}(A) = \left\{ \sum_{1\le i\le n} b_i \cdot row(A,i) : b_1,\ldots,b_n \in \mathsf{F}[\partial;\sigma,\delta] \right\},$$

the left module of the row space of A. The following lemma is shown analogously to (Storjohann, 1994, §4.3.1, Lemma 4).

Lemma 4.1. *Let $A \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ be nonsingular, with Hermite form H. Let $h_i = \deg_\partial H_{ii}$ for $1 \le i \le n$. For $v = (0,\ldots,0,v_\ell,\ldots,v_n) \in \mathsf{F}[\partial;\sigma,\delta]^{1\times n}$, with $\deg v_\ell < h_\ell$, then if $v \in \mathcal{L}(A)$ we have $v_\ell = 0$, and if $v_\ell \ne 0$ then $v \notin \mathcal{L}(A)$.*

The following Theorem is analogous to Storjohann (1994), §4.3.2, Lemma 5, with a different, slightly weaker degree bound.

Theorem 4.2. *Let $A \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ have full left row rank, with $\deg_\partial A_{ij} \le d$ for $1 \le i,j \le n$. Let $(d_1,\ldots,d_n)$ be a given vector of non-negative integers. Let T be an $n\times n$ matrix with $T_{ij} = \sum_{0\le k\le \varrho} t_{ijk}\partial^k$ for unknowns t_{ijk}, where $\varrho \ge (n-1)d + \max_i\{d_i - h_i\}$. Consider the system of equations in t_{ijk} with constraints:*

$$\begin{aligned}
&(TA)_{i,i,d_i} = 1, \text{ for } 1 \le i \le n, && \text{— diagonal entries are monic;}\\
&(TA)_{i,i,k} = 0, \text{ for } k > d_i, && \text{— diagonal entry in row } i \text{ has degree } d_i;\\
&(TA)_{i,j,k} = 0, \text{ for } i \ne j \text{ and } k \ge d_j && \text{— off diagonal entries have lower degree}\\
& && \quad \text{than diagonal entry in that column.}
\end{aligned} \tag{4.1}$$

By a solution for T *we mean an assignment of variables $t_{ijk} \leftarrow \alpha_{ijk} \in \mathsf{F}$ for some $1 \le i,j \le n$ and $0 \le k \le \varrho$ such* (4.1) *holds.*

Let $h_1,\ldots,h_n \in \mathsf{F}[\partial;\sigma,\delta]$ be the degrees of the diagonal entries of the Hermite form of A. The following statements about the above system hold:

(i) If $d_i \ge h_i$ for $1 \le i \le n$ then there exist a solution for T;

(ii) If there exists a positive integer $\ell \leq n$ such that $d_i = h_i$ for $1 \leq i < \ell$ and $d_\ell < h_\ell$ then there is no solution for T;

(iii) If $d_i = h_i$ for $1 \leq i \leq n$ then there is a unique solution for T such that $G = TA$ is equal to the Hermite form of A under that solution.

Proof. Let $H \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ be the Hermite form of A and let $U \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ be the unique unimodular matrix such that $UA = H$.

To show (i), let $D = \operatorname{diag}(\partial^{d_1-h_1}, \ldots, \partial^{d_n-h_n}) \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$, and consider the equality $DUA = DH$. Let $H^* \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ be the Hermite form of DH and $U^* \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ the unimodular matrix such that $U^*DUA = H^*$. We construct H^* from DH simply by reducing the entries above the diagonal (since it is already upper triangular). Thus U^* is upper triangular, with ones on the diagonal, and $\deg U^*_{ij} < (d_i - h_i) - (d_j - h_j)$ for $i < j$. We claim $T = U^*DU$ is a solution to (4.1) . First note that the particular choice of D, together with the definition of H^* ensure that the constraints of (4.1) are met. Furthermore, entries in the ith row of U^*D have degree at most $d_i - h_i$. By Theorem 3.8, $\deg U \leq (n-1)d$, whence $\deg T \leq (n-1)d + \max_i\{d_i - h_i\} \leq \varrho$.

To prove (ii), suppose by contradiction that there exists a nonnegative integer $\ell \leq n$ and a solution T such that $\deg_\partial((TA)_{ii}) = d_i$ for $1 \leq i < \ell$ and $\deg_\partial((TA)_{\ell\ell}) < h_\ell$. Note that $\operatorname{row}(TA,\ell) = ((TA)_{\ell,1}, \ldots, (TA)_{\ell,n})$ is in $\mathcal{L}(A)$. First, if $\ell = 1$, then $\deg(TA)_{\ell,1} < h_1$, so by Lemma 4.1, $(TA)_{1,1} = 0$, which is impossible since (4.1) ensures this entry is monic. Now assume $\ell > 1$. Then $\deg(TA)_{\ell,1} < h_1$ (to satisfy (4.1)), and hence by Lemma 4.1, so $(TA)_{\ell,1} = 0$. A simple induction shows that $(TA)_{\ell,j} = 0$ for $1 \leq j < \ell$. Now consider $(TA)_{\ell,\ell}$, which has degree $d_\ell < h_\ell$ by our assumption. Again by Lemma 4.1 $(TA)_{\ell,\ell} = 0$, which (4.1) ensures is monic, a contradiction.

If the conditions of (iii) hold, then by (i) there exists at least on solution for T. We can use an inductive proof similar to that used in our proof of (ii) to show that elements below the diagonal in TA are zero (i.e., that $(TA)_{ij} = 0$ for $i > j$). By the uniqueness of the Hermite form we must have $TA = H$. □

This theorem allows us to work with a partial order on the degree sequences. For any $(h_1,\ldots,h_n),(d_1,\ldots,d_n) \in \mathbb{Z}^n$, we say that $(h_1,\ldots,h_n) \preceq (d_1,\ldots,d_n)$ if and only if $h_i \leq d_i$ for all $1 \leq i \leq n$ (and similarly define $\precneqq$ for strict precedence). Thus, (4.1) has a solution if and only if $(h_1,\ldots,h_n) \preceq (d_1,\ldots,d_n)$ and this is unique if and only if $(h_1,\ldots,h_n) = (d_1,\ldots,h_n)$.

We now embed the system (4.1) into a system of linear equations over F, with no Ore component. We embed $\mathsf{F}[\partial;\sigma,\delta]$ into vectors over F via $\tau_\ell : \mathsf{F}[\partial;\sigma,\delta] \to \mathsf{F}^{\ell+1}$, with

$$\tau_\ell(u_0 + u_1\partial + u_2\partial^2 + \cdots + u_\ell\partial^{\ell-1}) = (u_0,\ldots,u_\ell) \in \mathsf{F}^{\ell+1}.$$

For $g \in \mathsf{F}[\partial;\sigma,\delta]$ of degree d, $u \in \mathsf{F}[\partial;\sigma,\delta]$ of degree at most m, and assuming $\ell \geq m+d$, the equation $ug = f$ can be realized by a matrix equation over F:

$$(u_0,\ldots,u_m) \underbrace{\begin{pmatrix} \tau_\ell(g) \\ \hline \tau_\ell(\partial g) \\ \hline \vdots \\ \hline \tau_\ell(\partial^m g) \end{pmatrix}}_{\mu^\ell_m(g) \in \mathsf{F}^{(m+1)\times(\ell+1)}} = (f_0,\ldots,f_\ell) \quad \Longleftrightarrow \quad \tau_m(u)\,\mu^\ell_m(g) = \tau_\ell(f).$$

Fixing $d_1, \ldots, d_n \in \mathbb{N}$ as in Theorem 4.2, and setting $\varrho \geq (n-1)d + \max_i\{d_i - h_i\}$, we can then study (4.1), as realized as (a subset of) the linear equations in the matrix equation over F:

$$\underbrace{\begin{pmatrix} \tau_\varrho(T_{11}) & \cdots & \tau_\varrho(T_{1n}) \\ \vdots & & \vdots \\ \tau_\varrho(T_{n1}) & \cdots & \tau_\varrho(T_{nn}) \end{pmatrix}}_{\widehat{T} \in \mathsf{F}^{n \times (\varrho+1)n}} \underbrace{\begin{pmatrix} \mu_\varrho^{\varrho+d}(A_{11}) & \cdots & \mu_\varrho^{\varrho+d}(A_{1n}) \\ \vdots & & \vdots \\ \mu_\varrho^{\varrho+d}(A_{n1}) & \cdots & \mu_\varrho^{\varrho+d}(A_{nn}) \end{pmatrix}}_{\widehat{A} \in \mathsf{F}^{n(\varrho+1) \times (\varrho+d+1)n}} = \underbrace{\begin{pmatrix} \tau_{\varrho+d}(G_{11}) & \cdots & \tau_{\varrho+d}(G_{1n}) \\ \vdots & & \vdots \\ \tau_{\varrho+d}(G_{n1}) & \cdots & \tau_{\varrho+d}(G_{nn}) \end{pmatrix}}_{\widehat{G} \in \mathsf{F}^{n \times (\varrho+d+1)n}}. \tag{4.2}$$

We can ignore those equations whose component from $\widehat{G}$ is unknown (i.e., not determined to be 1 or 0 as in Theorem 4.2) because each such equation introduces no constraint on the solution. Thus, by Theorem 4.2, if we know $d_1, \ldots, d_n$, this system will have a unique solution, from which we completely determine $\widehat{T}$.

Example 4.3. Consider the following matrix in $\mathbb{Q}(z)[\partial; \prime]$ (the differential polynomials over $\mathbb{Q}(z)$):

$$A = \begin{pmatrix} (z+1) + \partial & z + z\partial & \partial \\ (z^2+z) + z\partial & z+1 & 2\partial \\ (-z-z^2) - z\partial & z\partial & z\partial \end{pmatrix} \in \mathbb{Q}(z)[\partial; \prime]^{3\times 3}.$$

Assume for this example that we know the degrees of the entries in the Hermite form are $(d_1, d_2, d_3) = (1, 0, 2)$. Then $n = 3$, and we can set $\varrho = 2$, and have

$$\overbrace{\left(\begin{array}{ccc|ccc|ccc} t_{110} & t_{111} & t_{112} & t_{120} & t_{121} & t_{122} & t_{130} & t_{131} & t_{132} \\ \hline t_{210} & t_{211} & t_{212} & t_{220} & t_{221} & t_{222} & t_{230} & t_{231} & t_{232} \\ \hline t_{210} & t_{211} & t_{212} & t_{220} & t_{221} & t_{222} & t_{230} & t_{231} & t_{232} \end{array}\right)}^{\widehat{T}}$$

$$\overbrace{\left(\begin{array}{cccc|cccc|cccc} z+1 & 1 & 0 & 0 & z & z & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & z+1 & 1 & 0 & 1 & z+1 & z & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & z+1 & 1 & 0 & 2 & z+2 & z & 0 & 0 & 0 & 1 \\ \hline z^2+z & z & 0 & 0 & z+1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 2z+1 & z^2+z+1 & z & 0 & 1 & z+1 & 0 & 0 & 0 & 0 & 2 & 0 \\ 2 & 4z+2 & z^2+z+2 & z & 0 & 2 & z+1 & 0 & 0 & 0 & 0 & 2 \\ \hline -z^2-z & -z & 0 & 0 & 0 & z & 0 & 0 & 0 & z & 0 & 0 \\ -2z-1 & -z^2-z-1 & -z & 0 & 0 & 1 & z & 0 & 0 & 1 & z & 0 \\ -2 & -4z-2 & -z^2-z-2 & -z & 0 & 0 & 2 & z & 0 & 0 & 2 & z \end{array}\right)}^{\widehat{A}}$$

$$= \overbrace{\left(\begin{array}{ccc|ccc|cc} G_{110} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & G_{130} & G_{131} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & G_{230} & G_{231} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{330} & G_{331} & 1 & 0 \end{array}\right)}^{\widehat{G}}.$$

We can remove columns from $\widehat{A}$ for which there is an indeterminate in the corresponding column in $\widehat{G}$; these are coefficients of non-maximal degree in the column and are not specified in (4.1). Thus, we obtain the reduced system of equations

$$\overbrace{\left(\begin{array}{ccc|ccc|ccc} t_{110} & t_{111} & t_{112} & t_{120} & t_{121} & t_{122} & t_{130} & t_{131} & t_{132} \\ \hline t_{210} & t_{211} & t_{212} & t_{220} & t_{221} & t_{222} & t_{230} & t_{231} & t_{232} \\ \hline t_{210} & t_{211} & t_{212} & t_{220} & t_{221} & t_{222} & t_{230} & t_{231} & t_{232} \end{array}\right)}^{\widehat{T}}
\overbrace{\left(\begin{array}{ccc|cccc|cc} 1 & 0 & 0 & z & z & 0 & 0 & 0 & 0 \\ z+1 & 1 & 0 & 1 & z+1 & z & 0 & 1 & 0 \\ 2 & z+1 & 1 & 0 & 2 & z+2 & z & 0 & 1 \\ \hline z & 0 & 0 & z+1 & 0 & 0 & 0 & 0 & 0 \\ z^2+z+1 & z & 0 & 1 & z+1 & 0 & 0 & 2 & 0 \\ 4z+2 & z^2+z+2 & z & 0 & 2 & z+1 & 0 & 0 & 2 \\ \hline -z & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 \\ -z^2-z-1 & -z & 0 & 0 & 1 & z & 0 & z & 0 \\ -4z-2 & -z^2-z-2 & -z & 0 & 0 & 2 & z & 2 & z \end{array}\right)}^{\widetilde{A}}$$

$$= \overbrace{\left(\begin{array}{ccc|ccc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)}^{\widetilde{G}} \in \mathsf{F}^{n(\varrho+1)\times n(\varrho+1)}$$

and can solve for

$$\widehat{T} = \begin{pmatrix} \frac{z+1}{2\,z+1} & 0 & 0 & -\frac{z}{2\,z+1} & 0 & 0 & -\frac{z+1}{2\,z+1} & 0 & 0 \\ -\frac{z}{2\,z+1} & 0 & 0 & \frac{z+1}{2\,z+1} & 0 & 0 & \frac{z}{2\,z+1} & 0 & 0 \\ -\frac{2\,z^2+3\,z+2}{(z^2+z+2)(2\,z+1)} & -\frac{z}{z^2+z+2} & 0 & \frac{2\,z^2+z-1}{(z^2+z+2)(2\,z+1)} & \frac{z+1}{z^2+z+2} & 0 & \frac{2\,z^3-z^2-2\,z-1}{z(z^2+z+2)(2\,z+1)} & \frac{z}{z^2+z+2} & 0 \end{pmatrix}$$

which corresponds to

$$T = \begin{pmatrix} \frac{z+1}{2\,z+1} & -\frac{z}{2\,z+1} & -\frac{z+1}{2\,z+1} \\ -\frac{z}{2\,z+1} & \frac{z+1}{2\,z+1} & \frac{z}{2\,z+1} \\ -\frac{2\,z^2+3\,z+2}{(z^2+z+2)(2\,z+1)} - \frac{z}{z^2+z+2}\partial & \frac{2\,z^2+z-1}{(z^2+z+2)(2\,z+1)} + \frac{z+1}{z^2+z+2}\partial & \frac{2\,z^3-z^2-2\,z-1}{z(z^2+z+2)(2\,z+1)}) + \frac{z}{z^2+z+2}\partial \end{pmatrix} \in \mathbb{Q}(z)[\partial;\prime]^{3\times 3}$$

giving

$$H = TA = \begin{pmatrix} (z+1)+\partial & 0 & -\frac{z^2+2z-1}{2z+1}\partial \\ 0 & 1 & \frac{z^2+z+2}{2z+1}\partial \\ 0 & 0 & \frac{2z^3+3z^2-2z-5}{(z^2+z+2)(2z+1)}\partial + \partial^2 \end{pmatrix} \in \mathbb{Q}(z)[\partial;\prime]^{3\times 3}$$

in Hermite form.

We can now state our algorithm for computing the Hermite form give the degrees of the diagonal elements.

Algorithm `HermiteFormGivenDegrees`

Input: $A \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ of full row rank, with (unknown) Hermite form H with diagonal degrees $(h_1,\ldots,h_n) \in \mathbb{N}^n$;
Input: $(d_1,\ldots,d_n) \in \mathbb{N}^n$, the posited degrees of diagonal entries of H
Output: $H \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ if $(d_1,\ldots,d_n) = (h_1,\ldots,h_n)$, or a message that $(d_1,\ldots,d_n)$ is lexicographically smaller or larger than $(h_1,\ldots,h_n)$;

1: Let $\varrho = (n-1)d + \max_i d_i$
2: Form the matrix equation $\widehat{T}\widehat{A} = \widehat{G}$ as in (4.2)
3: Remove all columns from $\widehat{G}$ containing an indeterminate, and corresponding columns from $\widehat{A}$, to form the "reduced" linear system $\widehat{T}\widetilde{A} = \widetilde{G}$, where $\widetilde{A}$ and $\widetilde{G}$ are now matrices over F
4: **if** $\operatorname{rank}\widetilde{A} < (n+1)\varrho$ **then**
5: **return** "$(h_1,\ldots,h_n) \trianglelefteq\!\!\!\!\neq (d_1,\ldots,d_n)$" // System is underconstrained
6: **if** $\widehat{T}\widetilde{A} = \widetilde{G}$ has no solution **then**
7: **return** "$(h_1,\ldots,h_n) \not\trianglelefteq (d_1,\ldots,d_n)$" // System is inconsistent
8: Solve the system $\widehat{T}\widetilde{A} = \widetilde{G}$ for $\widehat{T}$
9: Construct $T \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ from $\widehat{T}$
10: **return** $H = TA$ and $U = T$

From Theorem 3.6 we know that each entry in the Hermite form of $A \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$, with $\deg_\partial A \le d$, has degree at most nd. If the diagonal entries of A have degrees $(h_1,\ldots,h_n)$, then we know that

$$(0,\ldots,0) \trianglelefteq (h_1,\ldots,h_n) \trianglelefteq (nd,nd,\ldots,nd).$$

Algorithm `HermiteFormGivenDegrees` detects whether our choice of degree sequence is equal to, larger than, or not larger than or equal to the actual one. Thus, a simple component-wise binary search allows us to find the actual degree sequence $(h_1,\ldots,h_n)$. That is, start by finding for the h_1 by executing `HermiteFormGivenDegrees` with degree sequence $(d_1,nd,\ldots,nd)$ for different values of d_1. This will require $O(\log(nd))$ attempts. Then search for h_2 using degree sequence $O(h_1,d_2,nd,\ldots,nd)$ for different values of d_2, etc. It will require at most $O(n\log(nd))$ attempts to find the entire correct degree sequence $(h_1,\ldots,h_n)$.

Lemma 4.4. *Given $A \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ of full row rank, where each entry has degree (in ∂) less than d, we can compute the Hermite form $H \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ of A, and $U \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ such that $UA = H$. The algorithm requires us to call* `HermiteFormGivenDegrees` *$O(n\log(nd))$ times, with input A and varying degree sequences.*

For a first, general analysis of the complexity we will assume that operations in F have unit cost (and hence no coefficient growth is accounted for). To perform the rank test in Step 4, the inconsistency test in Step 6, and the equation solution in Step 8, we can simply do an LU decomposition of $\widetilde{A}$ using gaussian elimination. $\widetilde{A}$ has size $n(\varrho+1)\times m$, where $n(\varrho+1) \le m \le n(\varrho+d+1)$, i.e., $O(n^2d)\times O(n^2d)$. Gaussian elimination can be accomplished with $O(n^6d^3)$ operations over any field F (even if F is non-commutative). Combining this with Lemma 4.4 we obtain the following.

Theorem 4.5. *Let $A \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ have full row rank with entries of degree (in ∂) less than d. We can compute the Hermite form $H \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ of A, and $U \in \mathsf{F}[\partial;\sigma,\delta]^{n\times n}$ such that $UA = H$. The algorithm requires $O(n^7 d^3 \log(nd))$ operations in F.*

We next analyze our algorithm for computing the Hermite form of a matrix $A \in \mathsf{k}(z)[\partial;\sigma,\delta]^{n\times n}$ over the field $\mathsf{F} = \mathsf{k}(z)$, where k is a field and z an indeterminate. Without loss of generality $A \in \mathsf{k}[z][\partial;\sigma,\delta]^{n\times n}$ by clearing denominators (which is a left-unimodular operation), but note that the Hermite form may still be in $\mathsf{k}(z)[\partial;\sigma,\delta]$ (see Example 4.3). We will also assume for convenience that $\sigma(z) \in \mathsf{k}[z]$ and $\deg \delta(z) \leq 1$. Thus $\partial z = \sigma(z)\partial + \delta(z) \in \mathsf{k}[z][\partial]$ and the degree in z and ∂ remains the unchanged. A more general analysis could follow similarly.

Multiplying two polynomials in $\mathsf{k}[z]$ of degree at most m can be accomplished with $O(\mathsf{M}(m))$ operations in k: $\mathsf{M}(m) = m^2$ using standard arithmetic or $\mathsf{M}(m) = m \log m \log\log m$ using fast arithmetic (Cantor and Kaltofen, 1991). We similarly assume that two integers with ℓ bits can be multiplied with $O(\mathsf{M}(l))$ bit operations.

Theorem 4.6. *Let $A \in \mathsf{k}[z][\partial;\sigma,\delta]^{n\times n}$ have full row rank with entries of degree at most d in ∂, and of degree at most e in z. Let $H \in \mathsf{k}(z)[\partial;\sigma,\delta]^{n\times n}$ be the Hermite form of A and $U \in \mathsf{k}(z)[\partial;\sigma,\delta]^{n\times n}$ such that $UA = H$.*

(a) $\deg_z H_{ij} \in O(n^2 de)$ *and* $\deg_z U_{ij} \in O(n^2 de)$ *for* $1 \leq i, j \leq n$.

(b) We can compute the Hermite form $H \in \mathsf{k}(z)[\partial;\sigma,\delta]^{n\times n}$ of A, and $U \in \mathsf{k}(z)[\partial;\sigma,\delta]^{n\times n}$ such that $UA = H$ with a deterministic algorithm that requires $O(n^7 d^3 \log(nd) \cdot \mathsf{M}(n^2 de))$ or $O\tilde{}(n^9 d^3 e)$ [†], operations in k.

(c) Assume k has at least $4n^2 de$ elements. We can compute the Hermite form $H \in \mathsf{k}(z)[\partial;\sigma,\delta]^{n\times n}$ of A, and $U \in \mathsf{k}(z)[\partial;\sigma,\delta]^{n\times n}$ such that $UA = H$ using an algorithm that requires an expected number $O(n^7 d^3 \log(nd) + n^7 d^3 e)$ of operations in k (using standard polynomial arithmetic). This algorithm is probabilistic of the Las Vegas type (never returning an incorrect answer).

Proof. To show (a), recall that the matrix $\widetilde{A} \in \mathsf{F}[z]$ is of size $O(n^2 d) \times O(n^2 d)$. Using Hadamard's bound and Cramer's rule, the numerators and denominators in $\widehat{T}$ thus have degree at most $n^2 de$ in z. $H = UA$ has the same degree bound in z.

To prove (b), we note that $n^2 de$ is also a bound on the degrees (in z) of any reasonable algorithm for computing $\widehat{T}$. For example, we could simply compute modulo an irreducible polynomial in $\mathsf{k}[z]$ of higher degree than twice Hadamard's bound, and recover the entries in $\mathsf{k}(z)$ by rational recovery. The stated total cost follows.

To show (c), we note that the tests for rank deficiency in Step 4, and inconsistency in Step 6, can be done by considering the equation $\widehat{T}\widetilde{A} = \widetilde{G} \bmod (z-\alpha)$ for a randomly chosen α from a subset of k of size at least $4n^2 de$. This follows because the largest invariant factor $w \in \mathsf{k}[z]$ of $\widetilde{A}$ has degree at most $n^2 de$ by Hadamard's bound (see part (a)), and the rank modulo $(z-\alpha)$ changes only if α is a root of w. By the Schwartz-Zippel Lemma (Schwartz, 1980) this happens with probability at most $1/4$ for each choice of α (and this probability of error can be made exponentially smaller by repeating with different random choices). Thus, these tests require only $O(n^6 d^3)$ operations in k to perform,

[†] We employ soft-Oh notation: for functions σ and φ we say $\sigma \in O\tilde{}(\varphi)$ if $\sigma \in O(\varphi \log^c \varphi)$ for some constant $c \geq 0$.

correctly with high probability. During the binary search for the degree sequence we only perform these cheaper tests, requiring a total of $O(n^7 d^3 \log(nd))$ operations in k before finding the correct degree sequence.

Once we have found the correct degree sequence, we employ Dixon's (1982) algorithm to solve the linear system over $\mathsf{k}(z)$ (this is the fastest known algorithm using standard matrix arithmetic, and is very effective in practice). This lifts the solution to the system modulo $(z-\alpha)^i$ for $i = 1, \ldots, 2n^2de$, where α is a non-root of the (unknown) largest invariant factor of A (i.e., is such that $\operatorname{rank} A = \operatorname{rank} A \bmod (z-\alpha)$). Computing the solution modulo $(z-\alpha)^{2n^2de}$ is sufficient to recover the solution in $\mathsf{k}(z)$ using rational function reconstruction, since both the numerator and denominator have degree less than n^2de by part (a); see (von zur Gathen and Gerhard, 2003), Section 5.7. A random choice of α from a subset of k of size $4n^2de$ is sufficient to obtain a non-zero of the largest invariant factor (and hence not change the dimension of the solution space) with probability at least $1/4$ by the Schwartz-Zippel Lemma. In the first step of Dixon's algorithm, we compute the LU-decomposition of A mod $(z-\alpha)$ using $O(n^6 d^3)$ operations in k. We then lift the solution to $\widehat{T}\widetilde{A} \equiv \widetilde{G} \bmod (z-\alpha)^i$ for $i = 0, \ldots, 2n^2de$. Each lifting step requires $O(n^5 d^2)$ operations in k, yielding a total cost of $O(n^7 d^3 e)$. □

Finally, we consider coefficient growth in $\mathbb{Q}$ of Ore polynomial rings over $\mathbb{Q}(z)$. For the computation, once we have constructed the matrix $\widetilde{A}$, we can bound the coefficient-sizes in $\widehat{T}$ directly using Hadamard-type bounds. We can then employ a Chinese remainder scheme to find the Hermite form using the above algorithm (or any other method, for that matter). For example, we could simply choose a single prime p with twice as many bits as the largest numerator or denominator in the solution to (4.2) and then compute modulo that prime, in $\mathbb{Z}_p[z]$; the rational coefficients of H can be recovered by integer rational reconstruction from their images in $\mathbb{Z}_p$ (von zur Gathen and Gerhard, 2003, §5.7).

However, for the purposes of analysis, it is interesting to see how big these coefficients can grow. We consider matrices in $A \in \mathbb{Z}[z][\partial;\sigma,\delta]^{n\times n}$ without loss of generality. For convenience in this analysis (though not in complete generality), we assume that $\deg_z \delta(z) \le 1$ and $\sigma(z) \in \mathbb{Z}[z]$, so $\partial z = \sigma(z)\partial + \delta(z) \in \mathbb{Z}[z]$.

For a polynomial $a = a_0 + a_1 z + \cdots + a_m z^m \in \mathbb{Z}[z]$, let $\|a\|_\infty = \max_i |a_i|$. For $f = f_0(z) + f_1(z)\partial + \cdots + f_r(z)\partial^r \in \mathbb{Z}[z][\partial;\sigma,\delta]$, let $\|f\|_\infty = \max_i \|f_i\|_\infty$. Define $\|A\|_\infty = \max_{ij} \|A_i j\|_\infty$. In equation (4.2), the entries in $\widetilde{A}$ have size at most

$$\|A\|_\infty^{(\varrho)} = \max_{ij} \max_{\ell} \{\|A_{ij}\|_\infty, \|\partial A_{ij}\|_\infty, \ldots, \|\partial^\varrho A_{ij}\|_\infty\} \in \mathbb{Z}. \tag{4.3}$$

Theorem 4.7. *Let $A \in \mathbb{Z}[z][\partial;\sigma,\delta]^{n\times n}$ be of full row rank and such that $\deg_\partial(A) = d$, $\deg_z(A) \le e$ and $\|A\|_\infty^{(\varrho)} \le \beta$. Then the Hermite form $H \in \mathbb{Q}(z)[\partial;\sigma,\delta]^{n\times n}$ and $U \in \mathbb{Q}(z)[\partial;\sigma,\delta]^{n\times n}$ such that $UA = H$ satisfy*

$$\log \|H\|_\infty,\ \log \|U\|_\infty \in O(n^2 d\,(\log e + \log \beta + \log n + \log d)).$$

Proof. Entries in $\widetilde{A}$ are polynomials in $\mathbb{Z}[z]$ of degree at most e and coefficient size at most β. Every minor of $\widetilde{A}$, and hence each entry in the solution $\widehat{T}$, is bounded by Hadamard's bound, which in this case is

$$\big((1+e)\beta(n^2 d)\big)^{O(n^2 d)}$$

(see Giesbrecht (1993) Theorem 1.5 for height bounds on polynomial products). □

By performing all computations modulo an appropriately large prime (as discussed above), we immediately get the following.

Corollary 4.8. *Let $A \in \mathbb{Z}[z][\partial;\sigma,\delta]^{n\times n}$ have full row rank with entries of degree at most d in ∂, of degree at most e in z, and $\|A\|_\infty^{(\varrho)} \leq \beta$ (where $\varrho = O(n^2 d)$). Let $H \in \mathbb{Q}(z)[\partial;\sigma,\delta]^{n\times n}$ be the Hermite form of A and $U \in \mathbb{Q}(z)[\partial;\sigma,\delta]^{n\times n}$ such that $UA = H$.*

We can compute the Hermite form $H \in \mathbb{Q}(z)[\partial;\sigma,\delta]^{n\times n}$ of A, and $U \in \mathbb{Q}(z)[\partial;\sigma,\delta]^{n\times n}$ such that $UA = H$, using an algorithm that requires an expected number $O((n^7 d^3 \log(nd) + n^7 d^3 e) \cdot M(n^2 d(\log e + \log\beta + \log n + \log d)))$, or $O^{\sim}(n^9 d^4 e \log\beta)$ bit operations. This algorithm is probabilistic of the Las Vegas type (never returning an incorrect answer).

The following corollary summarizes this growth explicitly over two common rings, the differential polynomials $\mathbb{Q}(z)[\partial;']$, and the shift polynomial $\mathbb{Q}(z)[\partial;\mathcal{S}]$.

Corollary 4.9. *Let $A \in \mathbb{Z}[z][\partial;\sigma,\delta]^{n\times n}$ be of full row rank and such that $\deg_\partial(A) = d$, $\deg_z(A) \leq e$, $H \in \mathbb{Q}(z)[\partial;\sigma,\delta]^{n\times n}$ the Hermite form of A, and $U \in \mathbb{Q}(z)[\partial;\sigma,\delta]^{n\times n}$ such that $UA = H$. For both the differential polynomials $\mathbb{Q}(z)[\partial;']$ (wjere $\sigma(z) = z$, $\delta(z) = 1$) and the shift polynomials $\mathbb{Q}(t)[\partial;\mathcal{S}]$ (where $\sigma(z) = z+1$, $\delta(z) = 0$), we have*

$$\log\|U\|_\infty,\ \log\|H\|_\infty \in O^{\sim}(n^2 d\,(e + \log\|A\|_\infty))$$

Proof. To show this for differential polynomials, we note that for $a = \sum_{0\leq i\leq d} a_i(z)\partial^i \in \mathbb{Z}[z][\partial;']$,

$$\partial^\ell a = \sum_{0\leq j\leq \ell} \binom{\ell}{j} \sum_{0\leq i\leq d} a_i(z)^{(j)} \partial^{\ell-j},$$

where $a_i(z)^{(j)}$ is the jth derivative of $a_i(z)$. Since only the first e derivatives of any a_i are non-zero

$$\|\partial^\ell a\|_\infty \leq \ell^e \cdot \|a\|_\infty \cdot e!$$

and hence $\log\|A\|_\infty^{(\varrho)} \in O(\log\|A\|_\infty + e\log(n^2 d))$ for $\varrho = O(n^2 d)$. The result follows by Theorem 4.7.

To show this for shift polynomials we note that for $a = \sum_{0\leq i\leq d} a_i(z)\partial^i \in \mathbb{Z}[\partial,\mathcal{S}]$,

$$\partial^\ell a = \sum_{0\leq i\leq d} a_i(z+\ell)\partial^i,$$

so

$$\|\partial^\ell a\|_\infty \leq \|a\|_\infty 2^{e/2}\ell^e,$$

and hence $\log\|A\|_\infty^{(\varrho)} \in O(\log\|A\|_\infty + e\log(n^2 d))$ for $\varrho = n^2 d$. Again, the result follows from Theorem 4.7. □

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